

Some applications of quasi-velocities in optimal control

Lígia Abrunheiro,¹ Margarida Camarinha,² José F. Cariñena,³ Jesús Clemente-Gallardo,⁴ Eduardo Martínez,⁵ and Patricia Santos⁶

¹*ISCA - Universidade de Aveiro, Rua Associação Humanitária dos Bombeiros de Aveiro, 3811-902 Aveiro - Portugal*

²*Departamento de Matematica, Universidade de Coimbra, Largo D. Dinis, Apartado 3008, 3001-454 Coimbra Portugal*

³*Departamento de Física Teórica Universidad de Zaragoza, 50009 Zaragoza, Spain*

⁴*BIFI-Departamento de Física Teórica Edificio I+D, Campus Río Ebro, Universidad de Zaragoza, C/Mariano Esquillor s/n, 50018 Zaragoza, Spain*

⁵*Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain*

⁶*CMUC- Universidade de Coimbra and Instituto Superior de Engenharia de Coimbra, Rua Pedro Nunes - Quinta da Nora 3030-199 Coimbra, Portugal*

Abstract

In this paper we study optimal control problems for nonholonomic systems defined on Lie algebroids by using quasi-velocities. We consider both kinematic, i.e. systems whose cost functional depends only on position and velocities, and dynamic optimal control problems, i.e. systems whose cost functional depends also on accelerations. The formulation of the problem directly at the level of Lie algebroids turns out to be the correct framework to explain in detail similar results appeared recently [20]. We also provide several examples to illustrate our construction.

I. INTRODUCTION

The principles of analytical mechanics established by D'Alembert, Lagrange, Gauss and Hamilton can also be contemplated from additional mathematical perspectives providing us methods for understanding Nature's law from new viewpoints which may be helpful in solving specific problems and clarifying the way in which Nature behaves. The traditional techniques were only applied to very simple models but current technology needs efficient algorithms in areas ranging from robotics to spacecraft design. Furthermore the computer development with the corresponding capability of computation suggests the convenience of analysing different formulations to yield the differential equations for multibody dynamics that involve a certain number of constraints.

There exist different techniques to deal with such constrained systems. The geometric framework of manifolds replacing Euclidean spaces allows us to give a formulation for systems with holonomic constraints in terms of generalised coordinates and free of Lagrange multipliers. However it is not clear how to choose generalised coordinates improving computational efficiency: Kane's method [14, 15] or the Maggi equations formulation [6]. Another recent alternative formulation is given in [13].

Nonholonomic constraints are very relevant and appear in many problems in physics and engineering, and in particular in control theory. Such nonholonomic constraints restrict possible virtual displacements and when taking into account such constraints d'Alembert-Lagrange principle leads to Boltzmann-Hamel equations [7, 9, 20–22].

The concept of quasi-velocity (or generalised velocity) [3] is of a great relevance in the study of mechanical systems, mainly in nonholonomic ones because the conditions of nonholonomic constraints can be expressed in a simpler form. Boltzmann-Hamel equations, Gibbs-Appell and Gauss principles, for instance, make use of quasi-coordinates (also called nonholonomic coordinates) and the Hamel symbols [16]. The use of quasi-velocities in the dynamics of nonholonomic system with symmetry has recently been investigated under different approaches in [2, 7–9] and Hamel's equations have been recovered from this perspective.

It has been shown in recent papers [3, 7, 8] that the appropriate geometric framework for studying systems with linear nonholonomic constraints is the framework of Lie algebroids. The geometric approach to mechanics uses tangent bundles in the Lagrangian formula-

tion and tangent bundles are but particular instances of algebroids. The usual geometric approach to Lagrangian formalism was then developed in this extended framework of Lie algebroids [5, 17, 18, 24] the main advantage being that such structure arises in reduction processes from tangent bundles when the vertical endomorphism character is not projectable [4]. The geometrical construction (see [17]) is based on the generalization of the usual symplectic description of Lagrangian (or Hamiltonian) mechanics on tangent (or cotangent) bundles. The dynamics is then defined by a function of a Lie algebroid (for the Lagrangian formalism) or its dual (for the Hamiltonian one). Considering, for the sake of simplicity, only the Lagrangian case now, the solutions of the dynamics correspond to the analogue of the concept of SODE (considered as the section of the bundle $T(TM)$ for a Lagrangian defined on TM). But in the Lie algebroid case the concept of second tangent bundle is subtle to define and the notion of prolongation of a Lie algebroid is necessary:

Definition I.1 *Let $(E, [\![\ , \]\!], \rho)$ be a Lie algebroid $(\tau : E \rightarrow M)$ over a manifold M and $\nu : P \rightarrow M$ be a fibration. For every point $p \in P$ we consider the vector space*

$$\mathcal{T}_p^E P = \{ (b, v) \in E_x \times T_p P \mid \rho(b) = T_p \nu(v) \},$$

where $T\nu : TP \rightarrow TM$ is the tangent map to ν and $\nu(p) = x$. The set $\mathcal{T}^E P = \bigcup_{p \in P} \mathcal{T}_p^E P$ has a natural vector bundle structure over P , the vector bundle projection τ_P^E being just the projection $\tau_P^E(b, v) = \tau_P(v)$.

The vector bundle $\tau_P^E : \mathcal{T}^E P \rightarrow P$ can be endowed with a Lie algebroid structure (see [17]). The Lie algebroid $\mathcal{T}^E P$ is called the **prolongation of $\nu : P \rightarrow M$ with respect to E** or the *E -tangent bundle to ν* .

When we consider the case $P = E$, the resulting extension generalizes the notion of second tangent bundle to the Lie algebroid framework and the sections of this bundle represent the analogue of SODEs in the usual case.

By using this notion, the symplectic formalism of Lagrangian systems in geometric mechanics is easily extended to this more general setting. In a similar way, Hamiltonian mechanics and discrete mechanics can be also generalized to the new framework.

It is also well known from geometric control theory that many of the techniques of classical mechanics can be used in control theory [7, 9, 19]. The theory of Lie algebroids can be applied to deal with control problems and the application of such geometric tools is very useful for

a better understanding of different control problems. This is our motivation for developing the theory of optimal control theory using the properties of Lie algebroid theory which is going to be the appropriate approach to Boltzmann–Hamel equations.

Throughout the paper, we consider a Lie algebroid $\tau : E \rightarrow M$ with anchor mapping $\rho : E \rightarrow TM$. When coordinates are required, we consider a local basis $\{x^i\}$ for the base manifold M , and a basis of sections $\{e_\alpha\}$ for the bundle E which provides a set of coordinates (x^i, y^α) for the Lie algebroid. The anchor mapping is then represented by the set of functions $\{\rho_i^\alpha\}$ and the Lie algebra structure by the structure functions $\{C_{\alpha\beta}^\gamma\}$. When considering the dual bundle E^* , the dual basis of sections $\{e^\alpha\}$ is chosen and the corresponding coordinates are denoted as (x^i, μ_α) . The corresponding exterior differential $d : \text{Sec } \bigwedge^k E^* \rightarrow \text{Sec } \bigwedge^{k+1} E^*$ defines the corresponding algebroid cohomology, with respect to which the concept of symplectic or presymplectic form can be defined. This concept will be used later on when providing the geometrical framework for the maximum principle.

Finally, we consider the coordinate functions for the extensions of the Lie algebroid E by a bundle P . We consider the general case although in practice we use only the case $P = E$ and the case $P = \mathcal{D} \subset E$, for \mathcal{D} a subbundle of the Lie algebroid. In any case, considering local coordinates (x^i, u^β) on P and a local basis $\{e_\alpha\}$ of sections of E , we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\beta\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \left(p, e_\alpha(\nu(p)), \rho_i^\alpha \frac{\partial}{\partial x^i} \Big|_p \right) \quad \text{and} \quad \mathcal{V}_\beta(p) = \left(p, 0, \frac{\partial}{\partial u^\beta} \Big|_p \right).$$

If $z = (p, b, v)$ is an element of $\mathcal{T}^E P$, with $b = z^\alpha e_\alpha$, then v is of the form $v = \rho_i^\alpha z^\alpha \frac{\partial}{\partial x^i} + v^\beta \frac{\partial}{\partial u^\beta}$, and we can write $z = z^\alpha \mathcal{X}_\alpha(p) + v^\beta \mathcal{V}_\beta(p)$. Vertical sections are linear combinations of $\{\mathcal{V}_\beta\}$. Analogously, when considering the dual object, $\mathcal{T}^E P^*$, we use a local basis $\{\mathcal{X}_\alpha, \mathcal{P}_\beta\}$, where $\{\mathcal{P}_\beta\}$ are the sections corresponding to the vertical elements $\mathcal{P}_\beta(p) = \left(p, 0, \frac{\partial}{\partial \nu_\beta} \Big|_p \right)$, for $\{(x^i, \nu_\beta)\}$ being a set of coordinates for the bundle P^* .

This article is organized in the following way. We address the interested reader to [7, 9, 19] for a detailed description of Lie algebroids and the construction of general optimal control systems. We provide a short review of those results in Sections II to IV while incorporating the formalism of quasi-velocities on them: Section II deals with the optimal control theory and the Pontryagin maximum principle [19], kinematic optimal control is studied in Section III and dynamical aspects are the aim on Section IV. The theory is illustrated in Section V with several examples.

II. OPTIMAL CONTROL THEORY

As it is well known, optimal control theory is a generalization of classical mechanics. The central result in the theory of optimal control systems is Pontryagin maximum principle. The reduction of optimal control problems can be performed within the framework of Lie algebroids, see [19]. This was done as in the case of classical mechanics, by introducing a general principle for any Lie algebroid and later studying the behavior under morphisms of Lie algebroids. See also [11] for a recent direct proof of Pontryagin principle in the context of general algebroids.

Pontryagin maximum principle [19]

By a control system on a Lie algebroid $\tau: E \rightarrow M$ with control space $\pi: B \rightarrow M$ we mean a section of $\sigma: B \rightarrow E$ along π . A trajectory of the system σ is an integral curve of the vector field $\rho \circ \sigma$ along π .

$$\begin{array}{ccc} & E & \xrightarrow{\rho} TM \\ \sigma \nearrow & \downarrow \tau & \nwarrow \tau_M \\ B & \xrightarrow{\pi} & M \end{array}$$

Given a cost function $L \in C^\infty(B)$ we want to minimize the integral of L over some set of trajectories of the system satisfying some boundary conditions. Then we define the Hamiltonian function $H \in C^\infty(E^* \times_M B)$ by $H(\mu, u) = \langle \mu, \sigma(u) \rangle - L(u)$ and the associated Hamiltonian control system σ_H (a section along $\text{pr}_1: E^* \times_M B \rightarrow E^*$ of $\mathcal{T}^E E^*$) defined on a subset of the manifold $E^* \times_M B$, by means of the symplectic equation

$$i_{\sigma_H} \Omega = dH, \quad (2.1)$$

where Ω is the canonical symplectic form defined on the bundle E (i.e. a section of the bundle $\bigwedge^2 E^*$ which is closed for the differential calculus defined on the Lie algebroid). The integral curves of the vector field $\rho(\sigma_H)$ are said to be the critical trajectories.

In the above expression, the meaning of i_{σ_H} is as follows: Let $\Phi: E \rightarrow E'$ be a morphism of the bundle $T \rightarrow M$ over a map $\varphi: M \rightarrow M'$ and let η be a section of E' along φ . If ω a section of $\bigwedge^p E'^*$ then $i_\eta \omega$ is the section of $\bigwedge^{p-1} E^*$ given by $(i_\eta \omega)_m(a_1, \dots, a_{p-1}) = \omega_{\varphi(m)}(\eta(m), \Phi(a_1), \dots, \Phi(a_{p-1}))$ for every $m \in M$ and $a_1, \dots, a_{p-1} \in E_m$. In our case, the

map Φ is the prolongation $\mathcal{T}\text{pr}_1: \mathcal{T}^E(E^* \times_M B) \rightarrow \mathcal{T}^E E^*$ of the map $\text{pr}_1: E^* \times_M B \rightarrow E^*$ (this last map fibered over the identity in M), and σ_H is a section along pr_1 . Therefore, $i_{\sigma_H}\Omega - dH$ is a section of the dual bundle to $\mathcal{T}^E(E^* \times_M B)$.

It is easy to see that the symplectic equation (2.1) has a unique solution defined on the following subset

$$S_H = \{ (\mu, u) \in E^* \times_M B \mid \langle dH(\mu, u), V \rangle = 0 \text{ for all } V \in \text{Ker } \mathcal{T}\text{pr}_1 \}.$$

Therefore, it is necessary to perform a stabilization constraint algorithm to determine the submanifold where integral curves of σ_H do exists.

In local coordinates, the solution to the above symplectic equation is

$$\sigma_H = \frac{\partial H}{\partial \mu_\alpha} \mathcal{X}_\alpha - \left[\rho_\alpha^i \frac{\partial H}{\partial x^i} + \mu_\gamma C_{\alpha\beta}^\gamma \frac{\partial H}{\partial \mu_\beta} \right] \mathcal{P}^\alpha,$$

defined on the subset where $\frac{\partial H}{\partial u^A} = 0$, and therefore the critical trajectories are the solution of the differential-algebraic equations

$$\dot{x}^i = \rho_\alpha^i \frac{\partial H}{\partial \mu_\alpha}; \quad \dot{\mu}_\alpha = - \left[\rho_\alpha^i \frac{\partial H}{\partial x^i} + \mu_\gamma C_{\alpha\beta}^\gamma \frac{\partial H}{\partial \mu_\beta} \right]; \quad 0 = \frac{\partial H}{\partial u^A}. \quad (2.2)$$

Notice that $\frac{\partial H}{\partial \mu_\alpha} = \sigma^\alpha$.

One can easily see that whenever it is possible to write $\mu_\alpha = p_i \rho_\alpha^i$ then the above differential equations reduce to the critical equations for the control system $Y = \rho(\sigma)$ on TM and the function L . Nevertheless it is not warranted that μ is of that form. For instance in the case of a Lie algebra, the anchor vanishes, $\rho = 0$, so that the factorization $\mu_\alpha = p_i \rho_\alpha^i$ will not be possible in general.

III. KINEMATIC OPTIMAL CONTROL

Let $\tau: E \rightarrow M$ be a Lie algebroid and \mathcal{D} a constraint distribution. Given a cost function $\kappa: E \rightarrow \mathbb{R}$, we consider the following kinematic optimal control problem: we can control directly all the (constrained) velocities, and we want to minimize some cost functional

$$I(a) = \int_\alpha^\beta \kappa(a(t)) dt,$$

for $a: [\alpha, \beta] \subset \mathbb{R} \rightarrow E$ over the set of admissible curves taking values in \mathcal{D} . We use coordinates (x^i, y^a) to denote the elements of this bundle, where y^a will represent the coordinates

with respect to some basis of section for \mathcal{D} , as in the last section. We use capital indices A, B, C, \dots to represent the coordinates $\{y^A, y^B, \dots\}$ corresponding to the elements of the fiber of E not contained in \mathcal{D} . Analogously, $\{\mu^A, \mu^B, \dots\}$ represent the coordinates for the fiber elements in E^* not corresponding to sections dual to the elements in \mathcal{D} .

Remark 1 *Whenever the cost function κ is a quadratic function defined on \mathcal{D} , the problem that we are considering is just the problem of sub-Riemannian geometry. In the case of a degree 1 homogeneous cost function this is sub-Finslerian geometry, and in the more general case this problem can be called sub-Lagrangian problem.*

Since we can control directly the velocities or pseudovelocities, the control bundle is $B = \mathcal{D}$ and the system map $\sigma : \mathcal{D} \rightarrow E$ is just the canonical inclusion $\sigma(a) = a$.

$$\begin{array}{ccc} & E & \xrightarrow{\rho} TM \\ \sigma \nearrow & \downarrow \tau & \nwarrow \tau_M \\ \mathcal{D} & \longrightarrow & M \end{array}$$

Pontryagin Hamiltonian is a function $H \in C^\infty(E^* \times_M B)$ defined as $H(\mu, b) = \langle \mu, \sigma(b) \rangle - \kappa(b)$, which in coordinates reads

$$H(x^i, \mu_a, \mu_A, u^a) = \mu_a u^a - \kappa(x^i, u^a). \quad (3.1)$$

The Maximum principle imposes the choice of the control functions such that

$$\mu_a = \frac{\partial \kappa}{\partial u^a} \quad \left(\text{from } \frac{\partial H}{\partial u^a} = 0 \right). \quad (3.2)$$

Under appropriate regularity conditions the set S_H of solutions of this equation is a submanifold of $E^* \times_M B$, which we call the critical submanifold. Frequently, this set is but the image of a section of $E^* \times_M B \rightarrow E^*$, locally given by

$$u^a = u^a(x, \mu). \quad (3.3)$$

Thus the set S_H is diffeomorphic to E^* and the optimal Hamiltonian, with local expression $H(x^i, \mu_a, u^a(x, \mu))$, defines via the canonical symplectic form a Hamiltonian system on E^* .

The restriction of the Pontryagin-Hamilton equations to this submanifold provides us with the control system

$$\dot{x}^i = \rho_a^i u^a \quad (3.4)$$

and the dynamics of the momenta

$$\dot{\mu}_a = - \left[\rho_a^i \frac{\partial H}{\partial x^i} + \mu_c C_{ab}^c u^b + \mu_B C_{ab}^B u^b \right] \quad (3.5)$$

$$\dot{\mu}_A = - \left[\rho_A^i \frac{\partial H}{\partial x^i} + \mu_c C_{Ab}^c u^b + \mu_B C_{Ab}^B u^b \right]. \quad (3.6)$$

These are the equations to be solved and to be used to determine, by using the mapping (3.3), the control functions defining the solution optimizing the value of the cost function. By substitution of $\mu_a = \frac{\partial \kappa}{\partial u^a}$ into these equations, and taking into account that $\frac{\partial H}{\partial x^i} = -\frac{\partial \kappa}{\partial x^i}$, we get

$$\begin{aligned} \dot{x}^i &= \rho_a^i u^a; & \frac{d}{dt} \left(\frac{\partial \kappa}{\partial u^a} \right) - \rho_a^i \frac{\partial \kappa}{\partial x^i} + \frac{\partial \kappa}{\partial u^c} C_{ab}^c u^b + \mu_B C_{ab}^B u^b &= 0; \\ \dot{\mu}_A + \mu_B C_{Ab}^B u^b - \rho_A^i \frac{\partial \kappa}{\partial x^i} + \frac{\partial \kappa}{\partial u^c} C_{Ab}^c u^b &= 0. \end{aligned} \quad (3.7)$$

These equations are also obtained in [20, 21] for the case $E = TM$.

Remark 2 *For simplicity we are considering only normal extremals. For abnormal extremals we just have to consider the Hamiltonian function to be $H = \mu_a u^a$ and solve the same equations, i.e.*

$$\mu_a = 0; \quad \dot{x}^i = \rho_a^i u^a; \quad \mu_B C_{ab}^B u^b = 0; \quad \dot{\mu}_A + \mu_B C_{Ab}^B u^b = 0. \quad (3.8)$$

with $(\mu_A(t)) \neq (0)$ for all t .

Remark 3 *In the particular case when $\mathcal{D} = E$ and $\sigma = \text{id}_E$ we recover the Euler-Lagrange equations on the Lie algebroid E for the Lagrangian $L = \kappa$. Also when $\mathcal{D} \subsetneq E$ we get the so-called vakonomic equations for the Lagrangian $L = \kappa$ (see [12])*

IV. DYNAMIC OPTIMAL CONTROL

In the dynamic problem, we can control directly the motion on a nonholonomic system, with the exception of the constraint forces, of course. For instance, we can consider the equations of motion to be $\delta L(z)|_{\mathcal{D}} = u$, with $u \in \mathcal{D}^*$ the control variables representing the external (generalized) forces acting on the system. Another possibility would be to consider systems on which the accelerations are the control variables.

In both kinds of problems the state space is the manifold \mathcal{D} and the control bundle $\pi: B \rightarrow \mathcal{D}$ is

$$B = \{ (z, \nu) \in \mathcal{T}^{\mathcal{D}}\mathcal{D} \times \mathcal{D}^* \mid z \in \text{Adm}(E) \text{ and } \delta L(z)|_{\mathcal{D}} = \nu \}. \quad (4.1)$$

An element (z, ν) of B is of the form $z = (a, a, v)$ with $a \in \mathcal{D}$ and $v \in T_a\mathcal{D}$ and where ν is determined by the equations $\langle \nu, b \rangle = \langle \delta L(z), b \rangle$ for every $b \in \mathcal{D}$.

When we consider the forces as control variables, since we are assuming that the constrained Lagrangian system is regular, we can identify B with $\text{pr}_1: \mathcal{D} \oplus \mathcal{D}^* \rightarrow \mathcal{D}$, via $(a, a, v; \nu) \equiv (a, \nu)$, because the vector v is determined by the point a and the equation $\delta L(z)|_{\mathcal{D}} = u$.

When we consider the accelerations as controls we can identify B with $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap \text{Adm}(E) \rightarrow \mathcal{D}$, via $(a, a, v; \nu) \equiv (a, a, v)$ because ν is determined by $\nu = \delta L(z)|_{\mathcal{D}}$.

From a formal point of view both problems are equivalent, since the relation between them is one-to-one and thus it is possible to use the optimal solution written in terms of accelerations to determine the optimal forces and viceversa. In other words, they are related by a feedback transformation.

However, from the practical point of view the second problem produces simpler expressions. Therefore, we can identify B with $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap \text{Adm}(E)$ and take coordinates (x^i, y^a, v^a) where v^a are the acceleration coordinates, i.e. our control variables.

On this set we also need to specify a control system where the optimization will be built. Such a system is specified by giving a section $\sigma: B \rightarrow \mathcal{T}^E\mathcal{D}$, i.e. the resulting system must always define an admissible velocity and acceleration. The Lie algebroid relevant for this case is the E -tangent to \mathcal{D} . An element of $\mathcal{T}^E\mathcal{D}$ is of the form $z = (a, b, w)$ with $a \in \mathcal{D}$, $b \in E$, with $\tau(a) = \tau(b)$ and $w \in T_a\mathcal{D}$ with $\rho(b) = T\tau(w)$. Taking a local basis $\{e_a\}$ for \mathcal{D} , and completing a local basis $\{e_a, e_A\}$ for E , we can write $a = y^a e_a$, $b = z^a e_a + z^A e_A$, and $w = (\rho_a^i z^a + \rho_A^i z^A) \frac{\partial}{\partial x^i} + w^a \frac{\partial}{\partial y^a}$. By taking coordinates (x^i) in the base, we have coordinates $(x^i, y^a, z^a, z^A, w^a)$ on $\mathcal{T}^E\mathcal{D}$. A local basis of sections of $\mathcal{T}^E\mathcal{D} \rightarrow \mathcal{D}$ is $\{\mathcal{X}_a, \mathcal{X}_A, \mathcal{V}_a\}$ and the element z can be written $z = z^a \mathcal{X}_a(x, y) + z^A \mathcal{X}_A(x, y) + w^a \mathcal{V}_a(x, y)$, and $\rho^1(z) = w = (\rho_a^i z^a + \rho_A^i z^A) \frac{\partial}{\partial x^i} + w^a \frac{\partial}{\partial y^a}$. The corresponding coordinates on the dual bundle $(\mathcal{T}^E\mathcal{D})^*$ will be denoted $(x^i, y^a, \mu_a, \mu_A, \pi_a)$.

If we choose to control the accelerations of the system $\{u^a\}$, the map $\sigma: B \rightarrow \mathcal{T}^E\mathcal{D}$ is

given by the natural inclusion $\sigma(z) = z$,

$$\begin{array}{ccc} & \mathcal{T}^E \mathcal{D} & \xrightarrow{\rho^1} T\mathcal{D} \\ & \downarrow \tau & \swarrow \tau_{\mathcal{D}} \\ B & \xrightarrow{\sigma} & \mathcal{D} \end{array}$$

which in coordinates corresponds to $\sigma(x^i, y^a, u^a) = (x^i, y^a, y^a, 0, u^a)$.

Given a cost function $\kappa : B \rightarrow \mathbb{R}$ we take the Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E \mathcal{D})^* \times B)$, defined as $H(\mu, z) = \langle \mu, \sigma(z) \rangle - \kappa(z)$ which in coordinates is

$$H(x^i, y^a, \mu_a, \mu_A, \pi_a, u^a) = \mu_a y^a + \pi_a u^a - \kappa(x^i, y^a, u^a),$$

where the control functions are the accelerations u^a .

From $\frac{\partial H}{\partial u^a} = 0$, we get

$$\pi_a = \frac{\partial \kappa}{\partial u^a}. \quad (4.2)$$

These equations determine the optimal submanifold S_H in $(\mathcal{T}^E \mathcal{D})^* \times_{\mathcal{D}} B$, which in this case is a section of the projection $(\mathcal{T}^E \mathcal{D})^* \times_{\mathcal{D}} B \rightarrow B$, locally given by the equations $u^a = u^a(x^i, y^a, \pi_a)$. On S_H , the equations of motion are the following. From $\dot{x} = \rho \frac{\partial H}{\partial \mu}$, since in this case the base variables are (x, y) , we get the original control system

$$\dot{x}^i = \rho_a^i y^a + \rho_A^i 0 = \rho_a^i y^a \quad \dot{y}^a = u^a. \quad (4.3)$$

Also, the equations of motion for π_a , written as $\dot{\pi}_a = -\frac{\partial H}{\partial y^a} = \frac{\partial \kappa}{\partial y^a} - \mu_a$, because all the structure functions involved vanish (i.e. \mathcal{V}_a commute with all the others). Therefore we get

$$\mu_a = \frac{\partial \kappa}{\partial y^a} - \frac{d}{dt} \left(\frac{\partial \kappa}{\partial u^a} \right) = \frac{\partial \kappa}{\partial y^a} - \dot{\pi}_a,$$

which is the combination that appear in [20], equation (11), for the case $E = TM$, under the notation κ_J and without a justification.

The equation of motion for μ_a is

$$-\dot{\mu}_a = \rho_a^i \frac{\partial H}{\partial x^i} + \mu_c C_{ab}^c y^b + \mu_C C_{ab}^C y^b = -\rho_a^i \frac{\partial \kappa}{\partial x^i} + \mu_c C_{ab}^c y^b + \mu_C C_{ab}^C y^b, \quad (4.4)$$

and the equation of motion for μ_A is

$$-\dot{\mu}_A = \rho_A^i \frac{\partial H}{\partial x^i} + \mu_c C_{Ab}^c y^b + \mu_C C_{Ab}^C y^b = -\rho_A^i \frac{\partial \kappa}{\partial x^i} + \mu_c C_{Ab}^c y^b + \mu_C C_{Ab}^C y^b. \quad (4.5)$$

These two last equations correspond to equations (18) and (19) obtained in [20] for the case $E = TM$.

Remark 4 *As in the previous case, we have considered only normal extremals. For abnormal extremals we just have to take the cost function $\kappa = 0$ and solve the same equations for $\kappa = 0$, that is*

$$\mu_a = 0; \quad \pi_a = 0; \quad \dot{x}^i = \rho_a^i y^a; \quad \mu_B C_{ab}^B y^b = 0; \quad \dot{\mu}_A + \mu_B C_{Ab}^B y^b = 0. \quad (4.6)$$

with $(\mu_A(t)) \neq (0)$ for all t . Interestingly, we get exactly the same equations (plus $\pi_a = 0$) as in the kinematic case.

V. EXAMPLES

First we discuss from the point of view of the theory of Lie algebroids an example studied in [21]. The result is naturally analogous to the results obtained there, but within the new framework the treatment of dynamical control systems becomes much more natural. After that, we study a few other examples of systems relevant for their applications and whose solutions, obtained in more involved ways, can be found in the literature.

A. Dynamic optimal control of the vertical rolling disc

For such a system, the state space manifold corresponds to $M = \mathbb{R}^2 \times S^1 \times S^1$, and we will use the coordinates $x = (x^1, x^2, x^3, x^4)$, where $x^1 = x$, $x^2 = y$, $x^3 = \theta$, and $x^4 = \phi$.

The rolling without slipping condition of the motion on the plane leads to a pair of nonholonomic constraints

$$\dot{x}^1 - \cos(x^4)\dot{x}^3 = 0 \quad \text{and} \quad \dot{x}^2 - \sin(x^4)\dot{x}^3 = 0.$$

We can define then a set of coordinates adapted to these constraints and write a set of coordinates $\{y\}$ for the new velocities. Thus the quasi-velocities correspond to:

$$y^1 = \dot{x}^1 - \cos(x^4)\dot{x}^3, \quad y^2 = \dot{x}^2 - \sin(x^4)\dot{x}^3, \quad y^3 = \dot{x}^3, \quad y^4 = \dot{x}^4.$$

Analogously the inverse transformation allows us to write:

$$\dot{x}^1 = y^1 + \cos(x^4)y^3, \quad \dot{x}^2 = y^2 + \sin(x^4)y^3, \quad \dot{x}^3 = y^3, \quad \dot{x}^4 = y^4.$$

The local basis of sections of TM determined by the quasi-velocities turns out to be:

$$e_1 = \frac{\partial}{\partial x^1}, \quad e_2 = \frac{\partial}{\partial x^2}, \quad e_3 = \cos(x^4)\frac{\partial}{\partial x^1} + \sin(x^4)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad e_4 = \frac{\partial}{\partial x^4}.$$

The Lie algebroid structure is the usual one for the tangent bundle. But in the basis above, the anchor mapping is written as:

$$\rho(e_1) = \frac{\partial}{\partial x^1}, \quad \rho(e_2) = \frac{\partial}{\partial x^2}, \quad \rho(e_3) = \cos(x^4) \frac{\partial}{\partial x^1} + \sin(x^4) \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad \rho(e_4) = \frac{\partial}{\partial x^4}.$$

The Lie algebra structure is obtained also as $[e_3, e_4] = \sin(x^4)e_1 - \cos(x^4)e_2$, all other elements being zero. We can read then the Hamel symbols $\gamma_{\alpha\beta}^\epsilon$ in [3, 20] .

In what regards the control part, we are considering a situation where we control the external forces in the directions of the admissible velocities. Thus, as the velocities on the constrained system are of the form

$$y^1 = y^2 = 0, \quad y^3 = \dot{x}^3 \quad \text{and} \quad y^4 = \dot{x}^4,$$

the natural coordinates are $\mathbf{a} = (x^1, x^2, x^3, x^4, y^3, y^4) = (x, y^a)$. The control bundle B becomes thus $\mathcal{D} \oplus \mathcal{D}^*$ and we take as coordinates $(x^i, y^3, y^4, u_3, u_4)$, where $u_3 = \frac{3}{2}\dot{y}^3$ and $u_4 = \frac{1}{4}\dot{y}^4$ for $u_a = (\delta L)_a$.

The cost function corresponds to $\kappa(\mathbf{a}, u) = \frac{1}{2}(u_3^2 + u_4^2)$ and the control system is defined as:

$$\dot{x}^1 = \cos(x^4)\dot{x}^3, \quad \dot{x}^2 = \sin(x^4)\dot{x}^3, \quad u_3 = \frac{3}{2} \frac{d^2 x^3}{dt^2}, \quad u_4 = \frac{1}{4} \frac{d^2 x^4}{dt^2}.$$

The Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E \mathcal{D})^* \times_{\mathcal{D}} B)$ corresponds now to

$$H(\mathbf{a}, p, u) = \langle p, \sigma_{\mathbf{a}}(u) \rangle - \kappa(\mathbf{a}, u) = \mu_I y^I + \pi_I \frac{u_I}{c_I} - \frac{1}{2}(u_3^2 + u_4^2),$$

with $c_3 = 3/2$ and $c_4 = 1/4$.

The Maximum principle is encoded as

$$\frac{\partial H}{\partial u_3} = 0 \Leftrightarrow u_3 = \frac{2}{3}\pi_3 \quad \text{and} \quad \frac{\partial H}{\partial u_4} = 0 \Leftrightarrow u_4 = 4\pi_4,$$

and the Pontryagin equations (optimal dynamical control equations) correspond to:

$$\dot{x}^1 = \cos(x^4)y^3, \quad \dot{x}^2 = \sin(x^4)y^3, \quad \dot{x}^3 = y^3, \quad \dot{x}^4 = y^4$$

$$\dot{y}^3 = \frac{2}{3}u_3, \quad \dot{y}^4 = 4u_4, \quad \dot{\pi}_3 = -\mu_3, \quad \dot{\pi}_4 = -\mu_4,$$

$$\dot{\mu}_1 = 0, \quad \dot{\mu}_2 = 0,$$

$$\dot{\mu}_3 = [-\mu_1 \sin(x^4) + \mu_2 \cos(x^4)]y^4, \quad \dot{\mu}_4 = [\mu_1 \sin(x^4) - \mu_2 \cos(x^4)]y^3.$$

Since $y^3 = \dot{x}^3$, $y^4 = \dot{x}^4$ and $\dot{y}^I = u_I$, then $\mu^3 = -\frac{9}{4}\frac{d^3x^3}{dt^3}$ and $\mu^4 = -\frac{1}{16}\frac{d^3x^4}{dt^3}$. Thus, we can reduce the set of equations to:

$$\begin{aligned}\dot{x}^1 &= \cos(x^4)\dot{x}^3, & \dot{x}^2 &= \sin(x^4)\dot{x}^4, \\ \frac{d^4x^3}{dt^4} &= \frac{4}{9}[\mu_1\sin(x^4) - \mu_2\cos(x^4)]\dot{x}^4, \\ \frac{d^4x^4}{dt^4} &= 16[-\mu_1\sin(x^4) + \mu_2\cos(x^4)]\dot{x}^3,\end{aligned}$$

where μ_1, μ_2 are constants.

B. Optimal control problems of rotational motion of the free rigid body

Consider the problem of rotational motion of the free rigid body. As configuration manifold we take the Lie group $SO(3)$ and choose the type-I Euler angles (x^1, x^2, x^3) as local coordinate system. We consider the canonical Lie algebroid structure of the tangent bundle $TSO(3)$, whose anchor map is $\rho = id_{TSO(3)}$. Let $\{e_1, e_2, e_3\}$ be the set of sections for the bundle

$$\begin{aligned}e_1 &= \sec(x^2)\sin(x^3)\frac{\partial}{\partial x^1} + \cos(x^3)\frac{\partial}{\partial x^2} + \tan(x^2)\sin(x^3)\frac{\partial}{\partial x^3}, \\ e_2 &= \sec(x^2)\cos(x^3)\frac{\partial}{\partial x^1} - \sin(x^3)\frac{\partial}{\partial x^2} + \tan(x^2)\cos(x^3)\frac{\partial}{\partial x^3}, \\ e_3 &= \frac{\partial}{\partial x^3},\end{aligned}$$

whose Lie algebra structure is determined by the relations $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$. The anchor and the Lie bracket are locally determined by the functions

$$\begin{aligned}\rho_1^1 &= \sec(x^2)\sin(x^3), & \rho_1^2 &= \cos(x^3), & \rho_1^3 &= \tan(x^2)\sin(x^3), \\ \rho_2^1 &= \sec(x^2)\cos(x^3), & \rho_2^2 &= -\sin(x^3), & \rho_2^3 &= \tan(x^2)\cos(x^3), \\ \rho_3^1 &= 0, & \rho_3^2 &= 0, & \rho_3^3 &= 1;\end{aligned}$$

and $C_{12}^3 = -C_{21}^3 = C_{23}^1 = -C_{32}^1 = C_{31}^2 = -C_{13}^2 = 1$.

We consider now the rigid body subject to the constraint $\dot{x}^1\cos(x^2)\sin(x^3) + \dot{x}^2\cos(x^3) = 0$. This implies that the constraint distribution \mathcal{D} is the 2-dimensional subbundle of $TSO(3)$ generated by e_2 and e_3 .

1. Constrained kinematic problem

Let us study the kinematic constrained system, i.e., the system with admissible velocities belonging to the subbundle $\mathcal{D} \subset TSO(3)$ defined by the condition $y^1 = 0$. Thus the system is

$$\dot{x}^1 = \sec(x^2)\cos(x^3)u^2, \quad \dot{x}^2 = -\sin(x^3)u^2, \quad \dot{x}^3 = \tan(x^2)\cos(x^3)u^2 + u^3, \quad (5.1)$$

The cost function corresponds to the energy provided by the controls $k(x^i, u^a) = \frac{1}{2} [I_2(u^2)^2 + I_3(u^3)^2]$. The Hamiltonian in this case is written as

$$H = \mu_2 u^2 + \mu_3 u^3 - \frac{1}{2} [I_2(u^2)^2 + I_3(u^3)^2].$$

Optimality conditions defining the submanifold S_H given in (3.2) are $\mu_2 = I_2 u^2; \mu_3 = I_3 u^3$. Using the representation $u^2 = u^2(x, \mu)$ and $u^3 = u^3(x, \mu)$ for S_H , the equations (3.5) become then the control system (5.1) together with

$$\dot{\mu}_2 + \mu_1 u^3 = 0, \quad \dot{\mu}_3 - \mu_1 u^2 = 0, \quad \dot{\mu}_1 + (I_3 - I_2)u^2 u^3 = 0.$$

In the case of the completely symmetric rigid body we get $\dot{\mu}_2 + \mu_1 u^3 = 0; \quad \dot{\mu}_3 - \mu_1 u^2 = 0$ and $\dot{\mu}_1 = 0$, which are equivalent to the equations obtained by Sastry and Montgomery in [23].

2. Constrained dynamic problem

Finally let us study the case of dynamic control for the constrained system. From the geometrical point of view, the control bundle B corresponds to $\mathcal{T}^D \mathcal{D} \cap \text{Adm}(TSO(3))$ and the system can be described as

$$\dot{x}^1 = \sec(x^2)\cos(x^3)y^2, \quad \dot{x}^2 = -\sin(x^3)y^2, \quad (5.2)$$

$$\dot{x}^3 = \tan(x^2)\cos(x^3)y^2 + y^3, \quad \dot{y}^2 = u^2, \quad \dot{y}^3 = u^3 \quad (5.3)$$

We consider now as cost function, the restriction to \mathcal{D} of the cost function

$$k(x^i, y^a, u^a) = \frac{1}{2} [(I_2)^2(u^2)^2 + (I_3)^2(u^3)^2 + (I_3 - I_2)^2(y^2)^2(y^3)^2],$$

where we assume that the control functions are the components of the admissible angular accelerations of our system $u^2 = v^2$ and $u^3 = v^3$.

Optimality condition leads to the submanifold W defined as $\pi_2 = (I_2)^2 u^2$ $\pi_3 = (I_3)^2 u^3$. Then W is defined by specifying $u^2 = u^2(x, y, \pi)$ and $u^3 = u^3(x, y, \pi)$. The motion on W corresponds then to the control system (5.2) and

$$\begin{aligned}\dot{\pi}_2 &= \frac{(M_1)^2}{y^2} - \mu_2, & \dot{\pi}_3 &= \frac{(M_1)^2}{y^3} - \mu_3, & \dot{\mu}_2 + \mu_1 y^3 &= 0, & \dot{\mu}_3 - \mu_1 y^2 &= 0, \\ \dot{\mu}_1 + \mu_3 y^2 - \mu_2 y^3 &= 0,\end{aligned}$$

where $M_1 = (I_3 - I_2)y^2 y^3$ is a torque on \mathcal{D} .

In the case of the completely symmetric rigid body we obtain

$$\dot{\pi}_2 = -\mu_2, \quad \dot{\pi}_3 = -\mu_3, \quad \dot{\mu}_2 + \mu_1 y^3 = 0, \quad \dot{\mu}_3 - \mu_1 y^2 = 0, \quad \dot{\mu}_1 + \mu_3 y^2 - \mu_2 y^3 = 0.$$

This system gives the following equations obtained by Crouch and Silva Leite in [10], Ex. 6.4, Case II

$$\frac{d^3 y^2}{dt^3} - \mu_1 y^3 = 0, \quad \frac{d^3 y^3}{dt^3} + \mu_1 y^2 = 0, \quad \dot{\mu}_1 - \frac{d^2 y^3}{dt^2} y^2 + \frac{d^2 y^2}{dt^2} y^3 = 0.$$

C. Systems with symmetry and constraints: quasi-coordinates for the Atiyah algebroid

Consider a ball rolling without sliding on a fixed table (see Example 8.12 in [8]). The configuration space is $Q = \mathbb{R}^2 \times SO(3)$, where $SO(3)$ is parameterized by the Eulerian angles θ, ϕ and ψ . In quasi-coordinates $(x, y, \theta, \phi, \psi, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ the energy may be expressed by $T = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)]$, where ω_x, ω_y and ω_z are the components of the angular velocity of the ball.

The system is invariant under $SO(3)$ transformations, and thus it is natural to consider the corresponding formulation on the Atiyah algebroid $E = TQ/SO(3) \equiv T\mathbb{R}^2 \times \mathbb{R}^3$. On that system we must still implement the nonholonomic constraint arising from the rolling-without-sliding conditions $\dot{x}^1 - r\omega_2 = 0$ and $\dot{x}^2 + r\omega_1 = 0$.

For the configuration space we can choose coordinates $M = Q/G = \mathbb{R}^2 \ni \mathbf{x} = (x^1, x^2)$, with $x^1 = x$ and $x^2 = y$. In what regards the fiber, we can choose thus a transformation mapping the set of fiber coordinates $\{\dot{x}^1, \dot{x}^2, \omega_3, \omega_1, \omega_2\}$ onto a new set $\{y^\alpha\}$. These quasi-velocities become then $y^i = \dot{x}^i$, $y^3 = \omega_3$, $y^4 = \dot{x}^1 - r\omega_2$, $y^5 = \dot{x}^2 + r\omega_1$.

Analogously we can consider the inverse transformation. Thus the original velocities can be written in terms of the quasi-velocities as

$$\dot{x}^i = y^i, \quad \omega_3 = y^3, \quad \omega_1 = -\frac{1}{r}y^2 + \frac{1}{r}y^5, \quad \omega_2 = \frac{1}{r}y^1 - \frac{1}{r}y^4$$

The local basis of sections of E determined by the quasi-velocities turns out to be

$$f_1 = e'_1 + \frac{1}{r}e'_4, \quad f_2 = e'_2 - \frac{1}{r}e'_3, \quad f_3 = e'_5, \quad f_4 = -\frac{1}{r}e'_4, \quad f_5 = \frac{1}{r}e'_3$$

where $\{e'_1, e'_2, \dots, e'_5\}$ is the local basis defined in [8], page 36.

With respect to this basis, the structure constants and the anchor mapping of the Lie algebroid structure become

$$[f_2, f_1] = [f_1, f_5] = [f_4, f_2] = [f_5, f_4] = \frac{1}{r^2}f_3,$$

$$[f_3, f_1] = [f_4, f_3] = f_5[f_2, f_3] = [f_3, f_5] = f_4,$$

$$\rho(f_1) = \partial_x, \quad \rho(f_2) = \partial_y,$$

the remaining elements being zero.

The set of admissible velocities becomes thus the fiber of the distribution \mathcal{D} , which corresponds to $\mathcal{D} = \{(x^i, y^\alpha) \in E \mid y^4 = y^5 = 0\}$. The coordinates for these points are therefore $a = (x^1, x^2, y^1, y^2, y^3) = (x^i, y^\alpha)$, where in terms of the original set of coordinates these correspond to $y^1 = \dot{x}^1 = u^1$, $y^2 = \dot{x}^2 = u^2$, $y^3 = \omega_3 = u^3$ and $y^4 = 0 = y^5$.

The dynamical system on the algebroid is defined by a Lagrangian function on E , which can be written in terms of the velocities as $L(x, y, \dot{x}, \dot{y}, \omega_1, \omega_2, \omega_3) = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + k^2(\omega_1^2 + \omega_2^2 + \omega_3^2)]$, and in terms of the quasi-velocities

$$L(x^i, y^\alpha) = \frac{1}{2} [(y^1)^2 + (y^2)^2 + k^2 r^{-2} ((y^1)^2 + (y^2)^2 + (y^4)^2 + (y^5)^2 - y^2 y^5 - y^1 y^4) + k^2 (y^3)^2]. \quad (5.4)$$

1. Kinematic Control Problem

Consider the following problem: determine the minimal value among the set of admissible solutions $a : \mathbb{R} \rightarrow E$, of the controlled Euler-Lagrange equations of the form $\delta L(a(t)) = 0$, $a(t) \in \mathcal{D}$, where the cost function is $\kappa(x^i, u^a) = \frac{1}{2} \left\{ (u^1)^2 + (u^2)^2 + \frac{k^2}{r^2} [(u^2)^2 + (u^1)^2] + k^2 (u^3)^2 \right\}$.

The control bundle is \mathcal{D} , and the section we consider $\sigma : (x^i, u^a) \in \mathcal{D} \rightarrow (x^i, u^a, 0, u^a) \in E$ is the canonical inclusion. Please notice that we use the notation u^a to denote the elements of the fiber of \mathcal{D} when considered as the control bundle.

The Pontryagin Hamiltonian is then written as a function $H \in C^\infty(E^* \times_{\mathbb{R}^2} \mathcal{D})$. The optimality conditions of the Maximum principle on this function imply

$$\frac{\partial H}{\partial u^a} = 0 \Leftrightarrow \mu_a = c_a u^a, \quad (5.5)$$

where $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$.

If we write the set of Pontryagin equations we see:

$$\begin{aligned} \dot{x}^i &= y^i, & \dot{\mu}_1 &= \mu_3 \frac{\mu_2}{c_2 r^2} + \mu_5 \frac{\mu_3}{c_3}, & \dot{\mu}_2 &= -\mu_4 \frac{\mu_3}{c_3} - \mu_3 \frac{\mu_1}{c_1 r^2}, \\ \dot{\mu}_3 &= -\mu_5 \frac{\mu_1}{c_1} + \mu_4 \frac{\mu_2}{c_2}, & \dot{\mu}_4 &= -\mu_3 \frac{\mu_2}{c_2 r^2} - \mu_5 \frac{\mu_3}{c_3}, & \dot{\mu}_5 &= \mu_4 \frac{\mu_3}{c_3} + \mu_3 \frac{\mu_1}{c_1 r^2}. \end{aligned}$$

Thus we can use (5.5) and the above equations to define the resulting system on \mathcal{D} :

$$\frac{d^2 x^1}{dt^2} = \frac{1}{c_1} (d_2 \omega_3 - \dot{x}^2 \omega_3), \quad \frac{d^2 x^2}{dt^2} = \frac{1}{c_2} (\dot{x}^1 \omega_3 - d_1 \omega_3), \quad \dot{\omega}_3 = \frac{1}{c_3} (d_1 \dot{x}^2 - d_2 \dot{x}^1),$$

with d_1, d_2 constants and $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$.

2. Dynamic Optimal Control Problem

Let us consider now a different control problem, where we are able to control the forces acting on the system, i.e. we consider a system corresponding to $(\delta L)_a = u_a$, where L is defined as (5.4) and δ represents the variational derivative. In this case, for the Lagrangian given above, this implies that $\dot{y}^a = \frac{u_a}{c_a}$, $a = 1, 2, 3$; where again $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$ (see equations 1.9.13 in [1]).

The control system is thus defined as a section $\sigma : (x^i, y^a, u_a) \in \mathcal{D} \oplus \mathcal{D}^* \mapsto (x^i, y^a, u_a, 0, u_a/c_a) \in \mathcal{T}^E \mathcal{D}$. The cost function now is the energy provided by the control functions: $\kappa(x^i, u_a) = \frac{1}{2} \sum_a u_a^2$. As a result, the Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E \mathcal{D})^* \times_{\mathcal{D}} B)$ reads now $H(x^i, y^a, \mu_a, \pi_a, u_a) = \mu_a y^a + \pi_a \frac{u_a}{c_a} - \frac{1}{2} \sum_a u_a^2$. The maximum principle applied to this function results

$$\frac{\partial H}{\partial u_a} = 0 \Leftrightarrow u_a = \frac{\pi_a}{c_a}, \text{ with } a = 1, 2, 3.$$

Then the optimal manifold corresponds to this submanifold of $(\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B$.

The Pontryagin equations on $(\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B$ are:

$$\begin{aligned}\dot{x}^i &= y^i, & \dot{y}^a &= \frac{u_a}{c_a}, & \dot{\pi}_a &= -\mu_a \\ \dot{\mu}_1 &= \mu_3 \frac{y^2}{r^2} + \mu_5 y^3, & \dot{\mu}_2 &= -\mu_4 y^3 - \mu_3 \frac{y^1}{r^2}, & \dot{\mu}_3 &= -\mu_5 y^1 + \mu_4 y^2, \\ \dot{\mu}_4 &= -\dot{\mu}_1, & \dot{\mu}_5 &= -\dot{\mu}_2.\end{aligned}$$

But if we restrict them to the optimized submanifold we obtain the reduced system:

$$\begin{aligned}\frac{d^4 x^1}{dt^4} &= \left[\frac{c_3}{c_1 r}\right]^2 \dot{x}^2 \frac{d^2 \omega}{dt^2} - \omega_3 \frac{d^3 x^2}{dt^3} - \frac{e_2}{c_1^2} \omega_3, \\ \frac{d^4 x^2}{dt^4} &= -\left[\frac{c_3}{c_2 r}\right]^2 \dot{x}^1 \frac{d^2 \omega}{dt^2} + \omega_3 \frac{d^3 x^1}{dt^3} + \frac{e_1}{c_2^2} \omega_3, \\ \frac{d^3 \omega_3}{dt^3} &= \left[\frac{c_2}{c_3}\right]^2 \dot{x}^1 \frac{d^3 x^2}{dt^3} - \left[\frac{c_1}{c_3}\right]^2 \dot{x}^2 \frac{d^3 x^1}{dt^3} + \frac{e_2}{c_3^2} \dot{x}^1 - \frac{e_1}{c_3^2} \dot{x}^2,\end{aligned}$$

where $c_1 = c_2 = 1 + k^2/r^2$, $c_3 = k^2$ and e_1, e_2 are arbitrary constants.

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